Operator space structure on Feichtinger's Segal algebra

Nico Spronk

Department of Pure Mathematics, University of Waterloo, Waterloo ON, N2L 3G1, Canada

In memory of my friend, Sean Crawford Andrew.

Abstract

We extend the definition, from the class of abelian groups to a general locally compact group G, of Feichtinger's remarkable Segal algebra $\mathrm{S}_0(G)$. In order to obtain functorial properties for non-abelian groups, in particular a tensor product formula, we endow $\mathrm{S}_0(G)$ with an operator space structure. With this structure $\mathrm{S}_0(G)$ is simultaneously an *operator Segal algebra* of the Fourier algebra $\mathrm{A}(G)$, and of the group algebra $\mathrm{L}^1(G)$. We show that this operator space structure is consistent with the major functorial properties: (i) $\mathrm{S}_0(G)\hat{\otimes}\mathrm{S}_0(H) \cong \mathrm{S}_0(G\times H)$ completely isomorphically (operator projective tensor product), if H is another locally compact group; (ii) the restriction map $u \mapsto u|_H : \mathrm{S}_0(G) \to \mathrm{S}_0(H)$ is completely surjective, if H is a closed subgroup; and (iii) $\tau_N : \mathrm{S}_0(G) \to \mathrm{S}_0(G/N)$ is completely surjective, where N is a normal subgroup and $\tau_N u(sN) = \int_N u(sn)dn$. We also show that $\mathrm{S}_0(G)$ is an invariant for G when it is treated simultaneously as a pointwise algebra and a convolutive algebra.

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Email address: `nspronk@math.uwaterloo.ca` (Nico Spronk).

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1 Introduction and Notation

1.1 History

In [7], Feichtinger defined, for any abelian group G, a Segal algebra $\mathrm{S}_0(G)$ of $\mathrm{L}^1(G)$. This Segal algebra is the minimal Segal algebra in $\mathrm{L}^1(G)$ which is closed under pointwise multiplication by characters and on which multiplication by any character is an isometry. It is proved in [7], that the Fourier transform induces an isomorphism $\mathrm{S}_0(G) \cong \mathrm{S}_0(\hat{G})$ where $\hat{G}$ is the dual group. Thus, we also have that $\mathrm{S}_0(G)$ is a Segal algebra in the Fourier algebra $\mathrm{A}(G)$ ($\cong \mathrm{L}^1(\hat{G})$), i.e. a dense ideal of $\mathrm{A}(G)$ which has a norm under which it is a Banach $\mathrm{A}(G)$-module. In fact, it is the minimal Segal algebra in $\mathrm{A}(G)$ which is translation invariant and on which translations are isometries.

For a general, not necessarily abelian, locally compact group G, the Fourier algebra is defined by Eymard [5]. There are hints in [6] of how to define $\mathrm{S}_0(G)$, as a Segal algebra in $\mathrm{A}(G)$. We develop this fully. $\mathrm{S}_0(G)$ is also a Segal algebra in the classical sense, i.e. a Segal algebra of $\mathrm{L}^1(G)$. We also develop, for general locally compact groups, the functorial properties which Feichtinger proved for abelian groups [7, Thm. 7]. One of Feichtinger's results is a tensor product formula: if G and H are locally compact abelian groups, then there is a natural isomorphism $\mathrm{S}_0(G) \otimes^\gamma \mathrm{S}_0(H) \cong \mathrm{S}_0(G \times H)$ (projective tensor product). For non-abelian G and H we cannot expect that $\mathrm{A}(G) \otimes^\gamma \mathrm{A}(H) \cong \mathrm{A}(G \times H)$, in general, by [18]. The theory of operator spaces, and the associated operator tensor product, allows us to obtain a satisfactory result from [3]: $\mathrm{A}(G) \hat{\otimes} \mathrm{A}(H) \cong \mathrm{A}(G \times H)$. Thus we are motivated to find a natural operator space structure on $\mathrm{S}_0(G)$, for a general locally compact G, which allows us to recover a tensor product formula. Analogous to the fact that $\mathrm{S}_0(G)$ does not have a fixed natural norm, but rather a family of equivalent norms, we find that our operator space structure is determined only up to complete isomorphism. In order to deal with our operator space structure, we find it more natural to deal with certain "dual" type matrices $\mathrm{T}_n(\mathcal{V})$ over an operator space $\mathcal{V}$, as opposed to the usual matrices $\mathrm{M}_n(\mathcal{V})$, which forces us to summarise a coherent theory of these in Section 1.3.

To underscore the naturality of our operator space structure we examine the other two major functorial properties, restriction to a closed subgroup and averaging over a closed normal subgroup. We show that our operator space structure is natural in the sense that it gives complete surjections, onto the Feichtinger algebra of the closed subgroup in the case of restriction, and onto the Feichtinger algebra of the quotient group in the case of averaging. See Sections 3.2 and 3.4.

In 3.5 we discuss an isomorphism theorem, characterising $\mathrm{S}_0(G)$ and an invariant of G. This makes no use of our operator space structure. As we note, our result is not even specific to $\mathrm{S}_0(G)$, but can be applied to many spaces which are simultaneously Segal algebras in $\mathrm{L}^1(G)$ and in some regular Banach subalgebra $\mathcal{A}$ of functions on G, having Gelfand spectrum G.

The Segal algebra $\mathrm{S}_0(G)$ seems interesting in and of itself simply for its wealth of structure and functorial properties. However, in the abelian case, $\mathrm{S}_0(G)$ is a fundamental example of *Wiener amalgam spaces* and of *modulation spaces*, which appear to be of tremendous use in time-frequency analysis. See [8] and references therein. We hope our $\mathrm{S}_0(G)$, for non-abelian G, may prove as inspirational and useful.

The author is grateful to Ebrahim Samei and Antoine Derighetti, for indicating to him the article [17], and helping to clarify some of the results therein. Though [17] was not ostensibly used, in the end, it inspired the author to find the proof of the result of Section 3.4.

1.2 Harmonic Analysis

Let G be a locally compact group with fixed left Haar measure m. We will denote integration of a function f with respect to m variously by $\int_G f dm$ or $\int_G f(s)ds$. For any $1 \leq p \leq \infty$ we let $\mathrm{L}^p(G)$ be the usual L^p-space with respect to m. If $s \in G$ and f is a complex-valued function on G we denote the group action of left translation of s on f by $s{*}f(t) = f(s^{-1}t)$ for t in G. For any appropriate pair of functions f, g we denote $f{*}g = \int_G f(s) s{*}g\, ds$.

The *Fourier* and *Fourier-Stieltjes algebras*, $\mathrm{A}(G)$ and $\mathrm{B}(G)$ were defined by Eymard in [5]. We recall that $\mathrm{A}(G)$ consists of functions on G of the form $s \mapsto \langle \lambda(s)f|g\rangle = \bar{g}{*}\check{f}(s)$, where $\lambda : G \to \mathcal{B}(\mathrm{L}^2(G))$ is the left regular representation where $\lambda(s)f(t) = f(s^{-1}t)$ for almost every t in G. We note that $\mathrm{A}(G)$ has Gelfand spectrum G, given by evaluation. We also remark that we have duality relations $\mathrm{A}(G)^* \cong \mathrm{VN}(G)$, where $\mathrm{VN}(G)$ is the von Neumann algebra generated by $\lambda(G)$, and $\mathrm{C}^*(G)^* \cong \mathrm{B}(G)$ where $\mathrm{C}^*(G)$ is the universal C*-algebra of G.

1.3 Operator Spaces

Our main reference for operator spaces is [4]. An operator space is a complex vector space $\mathcal{V}$, equipped with a sequence of norms, one on each space of $n{\times}n$ matrices over $\mathcal{V}$, $\mathrm{M}_n(\mathcal{V})$, which satisfy Ruan's axioms; we call this an *operator*

space structure. An operator space is complete if $\mathrm{M}_1(\mathcal{V})$ is complete, i.e. $\mathcal{V} = \mathrm{M}_1(\mathcal{V})$ is a Banach space. If $\mathcal{W}$ is another operator space and $T : \mathcal{V} \to \mathcal{W}$ is a linear map we let $T^{(n)} : \mathrm{M}_n(\mathcal{V}) \to \mathrm{M}_n(\mathcal{W})$ be the amplification given by $T^{(n)}[v_{ij}] = [Tv_{ij}]$. We say T is *completely bounded* if $\|T\|_{cb} = \sup_{n\in\mathbb{N}} \left\|T^{(n)}\right\| < \infty$. Moreover we say T is a *complete contraction/isometry/quotient map* if each $T^{(n)}$ is a contraction/isometry/quotient map. The space of completely bounded linear maps from $\mathcal{V}$ to $\mathcal{W}$ is denoted $\mathcal{CB}(\mathcal{V}, \mathcal{W})$.

We make note of some basic operator space constructions. If $\mathcal{W}$ is a closed subspace of an operator space $\mathcal{V}$, then $\mathcal{W}$ inherits the operator space structure from $\mathcal{V}$. Moreover, the quotient space $\mathcal{V}/\mathcal{W}$ obtains the *quotient* operator space structure via isometric identifications $\mathrm{M}_n(\mathcal{V}/\mathcal{W}) \cong \mathrm{M}_n(\mathcal{V})/\mathrm{M}_n(\mathcal{W})$, i.e.

$$\|[v_{ij} + \mathcal{W}]\|_{\mathrm{M}_n(\mathcal{V}/\mathcal{W})} = \inf\{\|[v_{ij} + w_{ij}]\|_{\mathrm{M}_n(\mathcal{V})} : [w_{ij}] \in \mathrm{M}_n(\mathcal{W})\}.$$

If $\mathcal{V}$ and $\mathcal{W}$ are any two operator spaces, then the space $\mathcal{CB}(\mathcal{V}, \mathcal{W})$ obtains the *standard* operator space structure (see [1]) where we identify, for each n, the matrix $[S_{ij}]$ in $\mathrm{M}_n(\mathcal{CB}(\mathcal{V}, \mathcal{W}))$ with the operator $v \mapsto [S_{ij}v]$ in $\mathcal{CB}(\mathcal{V}, \mathrm{M}_n(\mathcal{W}))$. We will make extensive use of the *operator projective tensor product*, defined in [3]. If $\mathcal{V}$ and $\mathcal{W}$ are two complete operator spaces, let $\mathcal{V}\hat{\otimes}\mathcal{W}$ denote their operator projective tensor product. The algebraic tensor product of $\mathcal{V}$ and $\mathcal{W}$ forms a dense subspace of $\mathcal{V}\hat{\otimes}\mathcal{W}$ and we let $\mathcal{V} \otimes_\wedge \mathcal{W}$ denote this algebraic tensor product, given the operator space projective structure.

We note that for any operator space $\mathcal{V}$ that every continuous linear functional is automatically completely bounded, i.e. $\mathcal{V}^* = \mathcal{CB}(\mathcal{V}, \mathbb{C})$, and thus is an operator space with the standard operator space structure. For a locally compact group G, the space $\mathrm{B}(G) \cong \mathrm{C}^*(G)^*$ is always endowed with the standard operator space structure. $\mathrm{A}(G)$ obtains the same operator space structure as a subspace of $\mathrm{B}(G)$ as it does as the predual of $\mathrm{VN}(G)$, i.e. a subspace of the dual. $\mathrm{L}^1(G)$, as the predual of a commutative von Neumann algebra naturally admits the *maximal* operator space structure.

Let $\mathcal{A}$ be a Banach algebra, equipped with an operator space structure such that $\mathrm{M}_1(\mathcal{A}) = \mathcal{A}$ isometrically, and let $\mathcal{V}$ be a left $\mathcal{A}$-module which is also an operator space. For a in $\mathcal{A}$ we let $m_a : \mathcal{V} \to \mathcal{V}$ be the module action map, $M_a v = a{\cdot}v$. Then $\mathcal{V}$ is called a *completely bounded Banach $\mathcal{A}$-module* under any of the following equivalent assumptions:

(i) $\{M_a : a \in \mathcal{A}\} \subset \mathcal{CB}(\mathcal{V})$ and $a \mapsto M_a : \mathcal{A} \to \mathcal{CB}(\mathcal{V})$ is completely bounded,

(ii) there is $C > 0$ such that for any pair of matrices $[a_{ij}]$ in $\mathrm{M}_n(\mathcal{A})$ and $[v_{kl}]$ in $\mathrm{M}_m(\mathcal{V})$, we have $\|[a_{ij}v_{kl}]\|_{\mathrm{M}_{nm}(\mathcal{V})} \leq C \|[a_{ij}]\|_{\mathrm{M}_n(\mathcal{A})} \|[v_{kl}]\|_{\mathrm{M}_m(\mathcal{V})}$, and

(iii) the map $\mathcal{A} \otimes_\wedge \mathcal{V} \to \mathcal{V}$, given on elementary tensors by $a \otimes v \mapsto a{\cdot}v$ is

completely bounded.

Similar definitions can be given with right and bimodules. We say $\mathcal{V}$ is a *completely contractive Banach $\mathcal{A}$-module* if the maps in (i) and (iii) above are complete contractions and in (ii) we can set $C = 1$. We call $\mathcal{A}$ a *completely bounded (contractive) Banach algebra*, if it is a completely bounded (contractive) module over itself. We note that $\mathrm{A}(G)$, $\mathrm{B}(G)$ and $\mathrm{L}^1(G)$ are all completely contractive Banach algebras with their standard operator space structures.

Operator Segal algebras were introduced in [9]. Let $\mathcal{A}$ be a completely contractive Banach algebra. An *operator Segal algebra* in $\mathcal{A}$ is a dense left ideal $\mathrm{S}\mathcal{A}$, equipped with a complete operator space structure such that

(OSA1) $\mathrm{S}\mathcal{A}$ is a completely bounded Banach $\mathcal{A}$-module, and

(OSA2) the injection $\mathrm{S}\mathcal{A} \hookrightarrow \mathcal{A}$ is completely bounded.

We further call $\mathrm{S}\mathcal{A}$ a *contractive operator Segal algebra* in $\mathcal{A}$ if the maps above are complete contractions. However, by uniformly scaling the matricial norms of $\mathrm{S}\mathcal{A}$ with a fixed small enough constant, we may make any operator Segal algebra a contractive one, and we will not insist on doing so in the sequel. We note that $\mathrm{S}\mathcal{A}$ itself is a completely bounded Banach algebra.

We finish this section by outlining an approach to operator spaces and completely bounded maps which is dual to the traditional one. Let, for the remainder of the section, $\mathcal{V}$ and $\mathcal{W}$ be complete operator spaces.

We let for any n in $\mathbb{N}$, T_n denote the operator space of $n \times n$ matrices with the dual operator space structure, i.e., $\mathrm{T}_n \cong \mathrm{M}_n^*$ completely isometrically. We let $\mathrm{T}_n(\mathcal{V}) = \mathrm{T}_n \hat{\otimes} \mathcal{V}$, which we regard as matrices. If $S : \mathcal{V} \to \mathcal{W}$ is a completely bounded map, then we let $\mathrm{T}_n(S) : \mathrm{T}_n(\mathcal{V}) \to \mathrm{T}_n(\mathcal{W})$ denote the amplification, i.e. $\mathrm{T}_n(S) = \mathrm{id}_{\mathrm{T}_n} \otimes S$. We also let T_∞ denote the set of $\mathbb{N} \times \mathbb{N}$ matrices which may be identified as trace class operators on $\ell^2(\mathbb{N})$, which we endow with the usual predual operator space structure $\mathrm{T}_\infty \cong \mathcal{B}(\ell^2(\mathbb{N}))_*$. We define $\mathrm{T}_\infty(\mathcal{V})$ analogously as above, and also the operator $\mathrm{T}_\infty(S)$, when it is defined. We note the following elementary, but important fact.

Proposition 1.1 *If $S : \mathcal{V} \to \mathcal{W}$ is a linear map, then the following are equivalent:*

(i) *S is a complete contraction [resp. complete quotient map],*

(ii) *each $\mathrm{T}_n(S)$ is a contraction [resp. quotient map], and*

(iii) *$\mathrm{T}_\infty(S)$ is defined and is a contraction [resp. quotient map].*

Proof. (i) $\Leftrightarrow$ (ii). This is [4, 4.18], in light of the identification [4, (7.1.90)] – which shows that our definition of $\mathrm{T}_n(\mathcal{V})$ coincides with theirs. If S is a complete quotient, then each $\mathrm{T}_n(S)$ is a quotient map by the projectivity property of the operator projective tensor product; see [4, 7.1.7].

(ii) $\Leftrightarrow$ (iii) for contractions. We have for each n a completely isomorphic embedding $\mathrm{T}_n \hookrightarrow \mathrm{T}_\infty$, given by identifying elements of T_n with elements of T_∞ whose non-zero entries are only in the upper left $n \times n$ corner. Since $\mathrm{T}_\infty^* \cong \mathcal{B}(\ell^2(\mathbb{N}))$ is an injective operator space, we obtain completely isometric imbeddings $\mathrm{T}_n(\mathcal{V}) = \mathrm{T}_n \hat{\otimes} \mathcal{V} \hookrightarrow \mathrm{T}_\infty(\mathcal{V}) = \mathrm{T}_\infty \hat{\otimes} \mathcal{V}$; see the discussion [4, p. 130]. As $\bigcup_{n=1}^\infty \mathrm{T}_n$ is dense in T_∞, it follows that $\mathrm{T}_{\mathrm{fin}}(\mathcal{V}) = \bigcup_{n=1}^\infty \mathrm{T}_n(\mathcal{V})$ is dense in $\mathrm{T}_\infty(\mathcal{V})$. Now we can define $\mathrm{T}_{\mathrm{fin}}(S) : \mathrm{T}_{\mathrm{fin}}(\mathcal{V}) \to \mathrm{T}_{\mathrm{fin}}(\mathcal{W})$ in the obvious way – so $\mathrm{T}_{\mathrm{fin}}(S) = \mathrm{T}_\infty(S)|_{\mathrm{T}_\infty(\mathcal{V})}$, when the latter makes sense. We have that

$$\|\mathrm{T}_{\mathrm{fin}}(S)\| = \sup_{n \in \mathbb{N}} \|\mathrm{T}_n(S)\| .$$

Thus if (ii) is assumed, then $\mathrm{T}_{\mathrm{fin}}(S)$ is contractive, and thus $\mathrm{T}_\infty(S)$ is defined and contractive. Conversely, if (iii) is assumed than $\mathrm{T}_{\mathrm{fin}}(S)$ is contractive,whence (ii).

(ii) $\Leftrightarrow$ (iii) for quotient maps. Suppose that $\mathrm{T}_\infty(S)$ is a quotient map. By [4, 10.1.4], we have that $\mathrm{T}_\infty(\mathcal{V})^* \cong \mathrm{M}_\infty(\mathcal{V}^*)$, where $\mathrm{M}_\infty(\mathcal{V}^*)$ is the space of $\mathbb{N} \times \mathbb{N}$ matrices with entries in $\mathcal{V}^*$ whose finite submatrices are uniformly bounded in norm. Thus $\mathrm{T}_\infty(S)^* = (S^*)^{(\infty)} : \mathrm{M}_\infty(\mathcal{W}^*) \to \mathrm{M}_\infty(\mathcal{V}^*)$ is an isometry. It follows that S^* is a complete isometry, whence S is a complete quotient map by [4, 4.1.8]. $\square$

We say that a linear operator $S : \mathcal{V} \to \mathcal{W}$ is a *complete isomorphism* if it is completely bounded, bijective, and $S^{-1} : \mathcal{W} \to \mathcal{V}$ is completely bounded too. We say that $S : \mathcal{V} \to \mathcal{W}$ is a *complete surjection* if the induced map $\tilde{S} : \mathcal{V}/\ker S \to \mathcal{W}$, defined by $\tilde{S}q = S$ where $q : \mathcal{V} \to \mathcal{V}/\ker S$ is the quotient map, is a complete isomorphism.

Corollary 1.2 (i) *Suppose S in $\mathcal{CB}(\mathcal{V}, \mathcal{W})$ is a bijection. Then S is a complete isomorphism if and only if $\mathrm{T}_\infty(S) : \mathrm{T}_\infty(\mathcal{V}) \to \mathrm{T}_\infty(\mathcal{W})$ is an isomorphism of Banach spaces.*

(ii) *Suppose S in $\mathcal{CB}(\mathcal{V}, \mathcal{W})$ is surjective. Then S is a complete surjection if and only if $\mathrm{T}_\infty(S) : \mathrm{T}_\infty(\mathcal{V}) \to \mathrm{T}_\infty(\mathcal{W})$ is surjective.*

Proof. (i) If $S^{-1} \in \mathcal{CB}(\mathcal{W}, \mathcal{V})$, then $\mathrm{T}_\infty(S^{-1}) = \mathrm{T}_\infty(S)^{-1}$. Conversely, if $T = \mathrm{T}_\infty(S)^{-1}$ is a bounded operator, we have that for w in $\mathrm{T}_{\mathrm{fin}}(\mathcal{W})$ that

$$\mathrm{T}_\infty(S)Tw = w = \mathrm{T}_\infty(S)\mathrm{T}_{\mathrm{fin}}(S^{-1})w$$

so $T|_{\mathrm{T}_{\mathrm{fin}}(\mathcal{W})} = \mathrm{T}_{\mathrm{fin}}(S^{-1})|_{\mathrm{T}_{\mathrm{fin}}(\mathcal{W})}$. Thus $\mathrm{T}_{\mathrm{fin}}(S^{-1})$ is bounded, whence so too is

$\mathrm{T}_\infty(S^{-1})$.

(ii) If S is surjective than $\tilde{S}$ is bijective. From above, if $\tilde{S}$ is a complete isomorphism, then $\mathrm{T}_\infty(\tilde{S})$ is an isomorphism of Banach spaces. It follows that $\mathrm{T}_\infty(S) = \mathrm{T}_\infty(\tilde{S})\mathrm{T}_\infty(q)$ is surjective. On the other hand, if $\mathrm{T}_\infty(S) = \mathrm{T}_\infty(\tilde{S})\mathrm{T}_\infty(q)$ is surjective then $\mathrm{T}_\infty(\tilde{S})$ is surjective. As it is already injective, and bounded as $\mathrm{T}_\infty(q)$ is a quotient map, we obtain that $\mathrm{T}_\infty(\tilde{S})$ is a bounded bijection, hence an isomorphism by the open mapping theorem. $\square$

We have, by [4, 7.1.6] that $\mathrm{T}_n\hat{\otimes}\mathcal{W}^* = \mathrm{T}_n(\mathcal{W}^*) \cong \mathrm{M}_n(\mathcal{W})^* \cong (\mathrm{M}_n\check{\otimes}\mathcal{W})*$, where the duality is given in tensor form by $\langle t\otimes f, m\otimes w\rangle = \mathrm{trace}(tm)f(w)$, for $t \in \mathrm{T}_n$, $f \in \mathcal{W}^*$, $m \in \mathrm{M}_n$ and $w \in \mathcal{W}$. Thus, in matrix form, this dual paring becomes

$$\Big\langle [f_{ij}], [w_{ij}] \Big\rangle = \sum_{i,j=1}^{n} f_{ij}(w_{ji})$$

for $[f_{ij}]$ in $\mathrm{T}_n(\mathcal{W}^*)$ and $[w_{ij}]$ in $\mathrm{M}_n(\mathcal{W})$. Thus the map $[S_{ij}]$ in $\mathcal{CB}(\mathcal{V}, \mathrm{M}_n(\mathcal{W}))$ has adjoint $[S_{ij}]^*$ in $\mathcal{CB}(\mathrm{T}_n(\mathcal{W}^*), \mathcal{V}^*)$ given by

$$\Big\langle [S_{ij}]^*[f_{ij}], v \Big\rangle = \Big\langle [f_{ij}], [S_{ij}v] \Big\rangle = \sum_{i,j=1}^{n} f_{ij}(S_{ji}v) = \sum_{i,j=1}^{n} S_{ji}^* f_{ij}(v)$$

for $[f_{ij}]$ in $\mathrm{T}_n(\mathcal{W}^*)$ and $v \in \mathcal{V}$.

Now if $[S_{ij}] \in \mathcal{CB}(\mathcal{V}, \mathrm{M}_n(\mathcal{W}))$, we have that $[S_{ij}^*] \in \mathcal{CB}(\mathcal{W}^*, \mathrm{M}_n(\mathcal{V}^*))$, with $\left\|[S_{ij}^*]\right\|_{cb} = \|[S_{ij}]\|_{cb}$, with proof similar to that of [4, 3.1.2]. Then $[S_{ij}^*]^* \in \mathcal{CB}(\mathrm{T}_n(\mathcal{V}^{**}), \mathcal{W}^{**})$ and we let

$$[S_{ij}^*]_* = [S_{ij}^*]^*|_{\mathrm{T}_n(\mathcal{V})} \in \mathcal{CB}(\mathrm{T}_n(\mathcal{V}), \mathcal{W}). \tag{1.1}$$

Thus if $[v_{ij}] \in \mathrm{T}_n(\mathcal{V})$, we have obtain $[S_{ij}^*]_*[v_{ij}] = \sum_{i,j=1}^{n} S_{ji}v_{ij}$. We have adjoint ${[S_{ij}^*]_*}^* = [S_{ij}^*]$, and hence

$$\left\|[S_{ij}^*]_*\right\|_{cb} = \left\|[S_{ij}^*]\right\|_{cb} = \|[S_{ij}]\|_{cb}\,. \tag{1.2}$$

This equation will be useful in the sequel when we determine that $\mathrm{S}_0(G)$ is an operator Segal algebra in $\mathrm{A}(G)$.

1.4 Localisation

Let $\mathcal{A}$ be a semi-simple commutative Banach algebra with Gelfand spectrum X. Via the Gelfand transform, we regard $\mathcal{A}$ as a subalgebra of $\mathcal{C}_0(X)$, the algebra of continuous functions on X vanishing at infinity. We say that $\mathcal{A}$ is *regular* if for every pair (x, F), where $x \in X$ and F is a closed subset of X

with $x \notin F$, we have that there is u in $\mathcal{A}$ such that $u(x) = 1$ and $u|_F = 0$. We note, below, that such an algebra admits local inverses.

Proposition 1.3 *If $\mathcal{A}$ is a regular Banach algebra with Gelfand spectrum X, $u \in \mathcal{A}$ and K is a compact subset of X on which u does not vanish, then there is u' in $\mathcal{A}$ such that $uu'|_K = 1$.*

Proof. By [13, (39.12)], $\mathcal{A}|_K = \{u|_K : u \in \mathcal{A}\}$ is a regular Banach algebra with Gelfand spectrum K. Then we may apply analytic functional calculus ([13, 39.14]) to see that $1/(u|_K) \in \mathcal{A}|_K$. Find v in $\mathcal{A}$ such that $v|_K = 1/(u|_K)$. $\square$

Thus, we see that a regular Banach algebra $\mathcal{A}$ is a "standard function algebra on its spectrum", in the sense of Reiter [22,23]. Let $\mathcal{I}$ be an ideal in $\mathcal{A}$, which is not necessarily assumed to be closed. We define the *hull* of $\mathcal{I}$ to be the set $\mathrm{h}(\mathcal{I}) = \{x \in X : u(x) = 0 \text{ for each } u \in \mathcal{I}\}$. We let $\mathcal{A}_c = \{u \in \mathcal{A} : \mathrm{supp} u \text{ is compact}\}$. Thus we obtain the following localisation result, proved in [22, 2.1.4] (or [23, 2.1.14]), which we restate here, without proof, for convenient reference.

Corollary 1.4 *Let $\mathcal{A}$ be a regular Banach algebra and $\mathcal{I}$ be an ideal of $\mathcal{A}$. If $u \in \mathcal{A}_c$ with $\mathrm{supp} u \cap \mathrm{h}(\mathcal{I}) = \varnothing$, then $u \in \mathcal{I}$. In particular, if $\mathrm{h}(\mathcal{I}) = \varnothing$, then $\mathcal{A}_c \subset \mathcal{I}$.*

We say that an ideal $\mathcal{I}$ of a regular Banach algebra $\mathcal{A}$ has *compact support*, if $\mathrm{supp}\mathcal{I} = \overline{G \setminus \mathrm{h}(\mathcal{I})}$ is compact in the spectrum X. The following result can be applied in more general situations than we give, but is only required for $\mathrm{A}(G)$ where G is a locally compact group.

Corollary 1.5 *Let $\mathcal{I}$ and $\mathcal{J}$ each be non-zero ideals having compact support in $\mathrm{A}(G)$ with $\mathcal{I} \subset \mathcal{J}$. Then there are $t_1, \dots, t_n$ in G and $u_1, \dots, u_n$ in $\mathcal{I}$ such that*

$$\sum_{l=1}^{n} (t_l * u_l) v = v \text{ for all } v \text{ in } \mathcal{J}.$$

Proof. Let Q be any compact subset of $\mathrm{supp}\mathcal{I}$ having non-empty interior Q°. Find $t_1, \dots, t_n$ in G so $\bigcup_{l=1}^n t_l Q^\circ \supset \mathrm{supp}\mathcal{J}$. Then $\mathcal{I}' = \sum_{l=1}^n t_l * \mathcal{I}$ is an ideal in $\mathrm{A}(G)$ for which $\mathrm{supp}\mathcal{J} \subset (\mathrm{supp}\mathcal{I})^\circ$. By the the regularity of $\mathrm{A}(G)$ and [13, (39.15)] (or see [5, (3.2)]), there is a function $u \in \mathrm{A}(G)$ such that $u|_{\mathrm{supp}\mathcal{J}} = 1$ and $\mathrm{supp} u \subset \bigcup_{l=1}^n t_l Q$. By Corollary 1.4, above, $u \in \mathcal{I}'$, hence $u = \sum_{l=1}^n t_l * u_l$, as desired. $\square$

2 Construction of Feichtinger's Segal Algebra

Let G be a locally compact group. In this section, we reconstruct Feichtinger's Segal algebra, which was done explicitly for abelian G in [7] and [23, §6.2] and, implicitly for general G, in [6]. The construction we give below is superficially different than the one of Feichtinger, but conceptually more useful to our task.

Let $\mathcal{I}$ be a non-zero closed ideal in $\mathrm{A}(G)$ which has compact support, as defined in Section 1.4. We let $\ell^1(G)$ be the usual ℓ^1-space, indexed over G, which will be identified with the closed linear span of the Dirac measures $\{\delta_s : s \in G\}$. We provide $\ell^1(G)$ with the maximal operator space structure. We define

$$q_{\mathcal{I}} : \ell^1(G)\hat{\otimes}\mathcal{I} \to \mathrm{A}(G) \quad \text{by} \quad q_{\mathcal{I}}(x \otimes u) = x*u = \sum_{s\in G} \alpha_s s*u$$

where $x = \sum_{s\in G} \alpha_s \delta_s$ and $s*u(t) = u(s^{-1}t)$ for t in G. We let

$$\mathrm{S}_0(G) = \operatorname{ran} q_{\mathcal{I}} \cong \ell^1(G)\hat{\otimes}\mathcal{I}/\ker q_{\mathcal{I}}.$$

We make $\mathrm{S}_0(G)$ into an operator space by giving it the quotient operator space structure. Let us note the following description of $\mathrm{S}_0(G)$, which is exactly that of [7], when $n = 1$ and $\mathcal{I} = \mathrm{A}_K(G)$, where

$$\mathrm{A}_K(G) = \{u \in \mathrm{A}(G) : \mathrm{supp} u \subset K\}$$

for K a compact subset of G with non-empty interior. The matrices $\mathrm{T}_n(\mathcal{V})$, for an operator space $\mathcal{V}$, are described in Section 1.3, above.

Lemma 2.1 *We have for any $n = 1, 2, \ldots, \infty$ that*

$$\mathrm{T}_n(\mathrm{S}_0(G)) = \left\{ \sum_{k=1}^{\infty} [s_k*u_{ij}^{(k)}] : \begin{array}{c} s_k \in G, [u_{ij}^{(k)}] \in \mathrm{T}_n(\mathcal{I}) \text{ with} \\ \sum_{i=1}^{\infty} \left\|[u_{ij}^{(k)}]\right\|_{\mathrm{T}_n(\mathrm{A})} < +\infty \end{array} \right\}. \tag{2.1}$$

The norm on $\mathrm{T}_n(\mathrm{S}_0(G)) = \mathrm{T}_n(\operatorname{ran} q_{\mathcal{I}})$ is given by

$$\|[u_{ij}]\|_{\mathrm{T}_n(\operatorname{ran} q_{\mathcal{I}})} = \inf\left\{ \sum_{k=1}^{\infty} \left\|[u_{ij}^{(k)}]\right\|_{\mathrm{T}_n(\mathrm{A})} : [u_{ij}] = \sum_{k=1}^{\infty} [s_k*u_{ij}^{(k)}] \text{ as above} \right\}.$$

Proof. First, let $n = 1$. We observe that there is an isometric identification $\ell^1(G)\hat{\otimes}\mathcal{I} = \ell^1(G) \otimes^\gamma \mathcal{I}$, by virtue of the fact that $\ell^1(G)$ has maximal operator space structure; see [4, 8.2.4]. Now if $t \in \ell^1(G) \otimes^\gamma \mathcal{I}$ and $\varepsilon > 0$, we can write

$$t = \sum_{k=1}^{\infty} \left(\sum_{s\in G} \alpha_{ks}\delta_s \right) \otimes u^{(k)}, \text{ where } \sum_{k=1}^{\infty} \left(\sum_{s\in G} |\alpha_{ks}| \right) \left\|u^{(k)}\right\|_{\mathrm{A}} < \|t\|_\gamma + \varepsilon.$$

We see the sum can easily be rearranged in the form $t = \sum_{s\in G} \delta_s \otimes \Big(\sum_{k=1}^{\infty} \alpha_{ks} u^{(k)}\Big)$. (This is essentially the proof that $\ell^1(G) \otimes^\gamma \mathcal{I}$ is isometrically isomorphic to the $\mathcal{I}$-valued ℓ^1-space, $\ell^1(G;\mathcal{I})$.) Thus we see that equations for u in $\mathrm{S}_0(G)$ can be arranged as suggested in (2.1), and any such equation describes an element of $\mathrm{S}_0(G) = \operatorname{ran} q_{\mathcal{I}}$. The formula for the norm follows immediately from the fact that $\mathrm{S}_0(G)$ is a quotient of $\ell^1(G) \otimes^\gamma \mathcal{I}$.

Now if $n > 1$, we make identifications

$$\mathrm{T}_n(\ell^1(G)\hat{\otimes}\mathcal{I}) \cong \mathrm{T}_n\hat{\otimes}\Big(\ell^1(G)\hat{\otimes}\mathcal{I}\Big) \cong \ell^1(G)\hat{\otimes}\Big(\mathrm{T}_n\hat{\otimes}\mathcal{I}\Big) = \ell^1(G) \otimes^\gamma \mathrm{T}_n(\mathcal{I}).$$

The proof then follows the $n = 1$ case, given above. $\square$

We now state some of the remarkable properties of $\mathrm{S}_0(G)$.

Theorem 2.2 **(i)** *The space* $\mathrm{S}_0(G)$ *is a (contractive) operator Segal algebra in* $\mathrm{A}(G)$*.*

(ii) $\mathrm{S}_0(G)$ *is the smallest Segal algebra* $\mathrm{SA}(G)$ *in* $\mathrm{A}(G)$ *which is closed under left translations and on which left translations are isometric, i.e.* $s{*}u \in \mathrm{SA}(G)$ *for each* s *in* G *and* u *in* $\mathrm{SA}(G)$ *with* $\|s{*}u\|_{\mathrm{SA}} = \|u\|_{\mathrm{SA}}$*. Moreover, for each* $u \in \mathrm{S}_0(G)$*,* $s \mapsto s{*}u : G \to \mathrm{S}_0(G)$ *is continuous, and for each* s *in* G*,* $u \mapsto s{*}u : \mathrm{S}_0(G) \to \mathrm{S}_0(G)$ *is a complete isometry.*

(iii) *For any closed ideals* $\mathcal{I}, \mathcal{J}$ *of* $\mathrm{A}(G)$*, each having compact support, the operator space structure on* $\mathrm{S}_0(G)$ *qua* $\operatorname{ran} q_{\mathcal{I}}$*, or on* $\mathrm{S}_0(G)$ *qua* $\operatorname{ran} q_{\mathcal{J}}$*, are completely isomorphic.*

Proof. **(i)** As in the construction above, we fix a non-zero closed ideal in $\mathrm{A}(G)$ with compact support.

We note that it is easy to see, using Lemma 2.1 above, and Proposition 1.1, that $\mathrm{S}_0(G)$ imbeds completely contractively into $\mathrm{A}(G)$. Moreover, using the $n = 1$ case of the lemma, it is easy to see that $\mathrm{S}_0(G)$ is a Banach $\mathrm{A}(G)$-module. Unfortunately, it is somewhat involved to show that $\mathrm{S}_0(G)$ is a completely contractive $\mathrm{A}(G)$-module.

Let for v in $\mathrm{A}(G)$, $M_v : \mathrm{A}(G) \to \mathrm{A}(G)$ be the multiplication map. Using (1.1) and (1.2), it suffices to show that for any $[v_{ij}] \in \mathrm{M}_n(\mathrm{A}(G))$ we have that

$$[M^*_{v_{ij}}]_* \in \mathcal{CB}(\mathrm{T}_n(\mathrm{S}_0(G)), \mathrm{S}_0(G)) \text{ with } \left\|[M^*_{v_{ij}}]_*\right\|_{cb} \leq \|[v_{ij}]\|_{\mathrm{M}_n(\mathrm{A})}.$$

By Proposition 1.1 it suffices to show for each m that

$$\mathrm{T}_m([M^*_{v_{ij}}]_*) : \mathrm{T}_m(\mathrm{T}_n(\mathrm{S}_0(G))) \to \mathrm{T}_m(\mathrm{S}_0(G))$$

satisfies

$$\left\| \mathrm{T}_m([M^*_{v_{ij}}]_*) \right\| \leq \|[v_{ij}]\|_{\mathrm{M}_n(\mathrm{A})} . \tag{2.2}$$

We note that there is a natural identification

$$\mathrm{T}_m(\mathrm{T}_n(\mathrm{S}_0(G))) \cong \mathrm{T}_{nm}(\mathrm{S}_0(G))$$

of which we take advantage. However, we will prefer, for computational convenience, to label elements of this space by doubly indexed matrices $[u_{ij,pq}]$, where $i, j = 1, \ldots, n$ and $p, q = 1, \ldots, m$.

By Lemma 2.1, each element $[u_{ij,pq}]$ of $\mathrm{T}_{nm}(\mathrm{S}_0(G))$ admits, for any $\varepsilon > 0$, the form

$$[u_{ij,pq}] = \sum_{k=1}^{\infty} [s_k * u^{(k)}_{ij,pq}] = \mathrm{T}_n(q_{\mathcal{I}}) \left(\sum_{k=1}^{\infty} \delta_{s_k} \otimes [u^{(k)}_{ij,pq}] \right)$$

where $\sum_{k=1}^{\infty} \left\| [u^{(k)}_{ij,pq}] \right\|_{\mathrm{T}_{nm}(\mathrm{A})} \leq \|[u_{ij}]\|_{\mathrm{T}_{nm}(\operatorname{ran} q_{\mathcal{I}})} + \varepsilon$. We see that

$$\begin{aligned}
\mathrm{T}_m([M^*_{v_{ij}}]_*)[w_{ij,pq}] &= \sum_{k=1}^{\infty} \mathrm{T}_m([M^*_{v_{ij}}]_*)\Big([s_k * u^{(k)}_{ij,pq}]\Big) \\
&= \sum_{k=1}^{\infty} \left[s_k * \sum_{i,j=1}^{m} (s_k^{-1} * v_{ji}) u_{ij,pq} \right] \\
&= \mathrm{T}_m(q_{\mathcal{I}}) \left(\sum_{k=1}^{\infty} \delta_{s_k} \otimes \left[\sum_{i,j=1}^{m} (s_k^{-1} * v_{ji}) u_{ij,pq} \right] \right) \\
&= \mathrm{T}_m(q_{\mathcal{I}}) \left(\sum_{k=1}^{\infty} \delta_{s_k} \otimes \mathrm{T}_m([M^*_{s_k^{-1} * v_{ij}}]_*)[u^{(k)}_{ij,pq}] \right) .
\end{aligned}$$

Using Proposition 1.1, (1.2), and the fact that translation is a complete isometry on $\mathrm{A}(G)$, we thus obtain

$$\begin{aligned}
&\left\| \sum_{k=1}^{\infty} \delta_{s_k} \otimes \mathrm{T}_m([M^*_{s_k^{-1} * v_{ij}}]_*)[u^{(k)}_{ij,pq}] \right\|_{\mathrm{T}_{nm}(\ell^1 \hat{\otimes} \mathcal{I})} \\
&\qquad \leq \sum_{k=1}^{\infty} \left\| \mathrm{T}_m([M^*_{s_k^{-1} * v_{ij}}]_*) \right\| \left\| [u^{(k)}_{ij,pq}] \right\|_{\mathrm{T}_{nm}(\mathrm{A})} \\
&\qquad = \sum_{k=1}^{\infty} \left\| [s_k^{-1} * v_{ij}] \right\|_{\mathrm{M}_m(\mathrm{A})} \left\| [u^{(k)}_{ij,pq}] \right\|_{\mathrm{T}_{nm}(\mathrm{A})} \\
&\qquad \leq \|[v_{ij}]\|_{\mathrm{M}_n(\mathrm{A})} \Big(\|[u_{ij,pq}]\|_{\mathrm{T}_{nm}(\operatorname{ran} q_{\mathcal{I}})} + \varepsilon \Big).
\end{aligned}$$

Thus, since ε is arbitrary, we obtain that

$$\left\| \mathrm{T}_m([M^*_{v_{ij}}]_*)[u_{ij,pq}] \right\|_{\mathrm{T}_m(\operatorname{ran} q_{\mathcal{I}})} \leq \|[v_{ij}]\|_{\mathrm{M}_n(\mathrm{A})} \|[u_{ij,pq}]\|_{\mathrm{T}_{nm}(\operatorname{ran} q_{\mathcal{I}})}$$

and hence we obtain (2.2).

(ii) It it immediate from the construction of $\mathrm{S}_0(G)$ that it is closed under left translations. We note that the action of G on $\mathrm{S}_0(G)$ is continuous, and is one of isometries, in fact complete isometries. This follows by a straightforward application of Lemma 2.1 and Proposition 1.1.

Any Segal algebra $\mathrm{SA}(G)$ in $\mathrm{A}(G)$ is an ideal with empty hull, and thus, by Corollary 1.4, necessarily contains $\mathrm{A}_c(G)$. Hence for any non-zero compactly supported ideal $\mathcal{I}$ of $\mathrm{A}(G)$, we obtain that $\mathcal{I} \subset \mathrm{SA}(G)$. Thus, if translations are pointwise continuous and isometric on $\mathrm{SA}(G)$, we see by Lemma 2.1, in the case $n = 1$, that $\mathrm{SA}(G) \supset \mathrm{S}_0(G)$.

(iii) It follows from (ii), above, that $\mathrm{S}_0(G)$ is independent if the choice of ideal $\mathcal{I}$. Let us suppose that $\mathcal{I}$ and $\mathcal{J}$ are two closed non-zero ideals of $\mathrm{A}(G)$ having compact supports. By replacing $\mathcal{I}$ by $\mathcal{I} \cap \mathcal{J}$, if necessary, we may suppose $\mathcal{I} \subset \mathcal{J}$. Then the injection $\iota : \ell^1(G)\hat{\otimes}\mathcal{I} \hookrightarrow \ell^1(G)\hat{\otimes}\mathcal{J}$ is a complete contraction. It is clear that $q_{\mathcal{J}}\circ\iota = q_{\mathcal{I}}$, so ι induces a completely contractive map $\tilde{\iota} : \operatorname{ran} q_{\mathcal{I}} \to \operatorname{ran} q_{\mathcal{J}}$, which is the identity map. Thus, by Proposition 1.1, $\mathrm{T}_\infty(\tilde{\iota}) : \mathrm{T}_\infty(\operatorname{ran} q_{\mathcal{I}}) \to \mathrm{T}_\infty(\operatorname{ran} q_{\mathcal{J}})$ is a contraction. Let us see that $\mathrm{T}_\infty(\tilde{\iota})$ is surjective. Let $u = \sum_{l=1}^n t_l * u_l$ be as in Corollary 1.5. We note that if $[w_{ij}] \in \mathrm{T}_\infty(\operatorname{ran} q_{\mathcal{J}})$, has form $[w_{ij}] = \sum_{k=1}^\infty [s_k * w_{ij}^{(k)}]$ as in (2.1), then we have

$$\begin{aligned}[w_{ij}] &= \sum_{k=1}^{\infty}[s_k*(uw_{ij}^{(k)})] = \sum_{k=1}^{\infty}\sum_{l=1}^{n}[s_k*(t_l*u_l\, w_{ij}^{(k)})] \\ &= \mathrm{T}_\infty(q_{\mathcal{I}})\left(\sum_{k=1}^{\infty}\sum_{l=1}^{n}\delta_{s_k t_l}\otimes[u_l(t_l^{-1}*w_{ij}^{(k)})]\right)\end{aligned} \tag{2.3}$$

which is an element of $\mathrm{T}_\infty(\operatorname{ran} q_{\mathcal{I}})$. Hence $\mathrm{T}_\infty(\tilde{\iota})$ is surjective, and thus, by the open mapping theorem, an isomorphism of Banach spaces. We then appeal to Corollary 1.2. □

Let us note that Theorem 2.2, above, holds under more general assumptions. Though the assumptions we give below seem less natural, they are indispensable for actually working with $\mathrm{S}_0(G)$. Let $\mathcal{I}$ be a non-zero compactly supported ideal in $\mathrm{A}(G)$, which is not necessarily closed, but comes equipped with an operator space structure by which it is a completely contractive Banach $\mathrm{A}(G)$-module, and the inclusion map $\mathcal{I} \hookrightarrow \mathrm{A}(G)$ is completely bounded. We define $q_{\mathcal{I}} : \ell^1(G)\hat{\otimes}\mathcal{I} \to \mathrm{A}(G)$, and its quotient space $\operatorname{ran} q_{\mathcal{I}}$, with its quotient operator space structure, as before.

Corollary 2.3 *With $\mathcal{I}$ as above, $\operatorname{ran} q_{\mathcal{I}}$ is an operator Segal algebra in $\mathrm{A}(G)$ which is completely isomorphic with $\operatorname{ran} q_{\bar{\mathcal{I}}}$, where $\bar{\mathcal{I}}$ is the closure of $\mathcal{I}$ in $\mathrm{A}(G)$. Hence $\operatorname{ran} q_{\mathcal{I}} = \mathrm{S}_0(G)$, completely isomorphically.*

Proof. We first note that the proofs of Lemma 2.1, and then of (i) and (ii) of the theorem above, can be applied verbatim; though we should note that

the inclusion $\operatorname{ran} q_{\mathcal{I}} \hookrightarrow \mathrm{A}(G)$ is completely bounded, instead of completely contractive. Thus we see that $\operatorname{ran} q_{\mathcal{I}}$ is an operator Segal algebra in $\mathrm{S}_0(G)$ on which G acts continuously and isomorphically by translations. Hence $\operatorname{ran} q_{\mathcal{I}} = \operatorname{ran} q_{\bar{\mathcal{I}}}$. Moreover the proof of part (iii) can be applied up to seeing that $\mathrm{T}_\infty(\tilde{\iota}) : \mathrm{T}_\infty(\operatorname{ran} q_{\mathcal{I}}) \to \mathrm{T}_\infty(\operatorname{ran} q_{\bar{\mathcal{I}}})$ is bounded. To see obtain surjectivity, we note for any u in $\mathcal{I}$ that $M_u : \mathrm{A}(G) \to \mathcal{I}$ is completely bounded, and then (2.3), where $[w_{ij}] \in \mathrm{T}_\infty(\operatorname{ran} q_{\bar{\mathcal{I}}})$, has the form

$$[w_{ij}] = \mathrm{T}_\infty(q_{\mathcal{I}}) \left(\sum_{k=1}^{\infty} \sum_{l=1}^{n} \delta_{s_k t_l} \otimes \mathrm{T}_\infty(M_{u_l})[t_l^{-1} * w_{ij}^{(k)}] \right)$$

where each $\mathrm{T}_\infty(M_{u_l})[t_l^{-1} * w_{ij}^{(k)}] \in \mathrm{T}_\infty(\mathcal{I})$. □

We recall that a Segal algebra $\mathrm{S}^1(G)$ in $\mathrm{L}^1(G)$ is called *symmetric* if for any s in G and f in $\mathrm{S}^1(G)$ we have

$$\|s*f\|_{\mathrm{S}^1} = \|f\|_{\mathrm{S}^1} = \|f*s\|_{\mathrm{S}^1}$$

where $f*s(t) = \Delta(s)^{-1} f(ts^{-1})$ for almost every t in G. This is a necessary and sufficient condition to make $\mathrm{S}^1(G)$ a two-sided ideal in $\mathrm{L}^1(G)$. $\mathrm{S}^1(G)$ is called *pseudo-symmetric* if the anti-action $s \mapsto f*s$ is continuous on G for any fixed f in $\mathrm{S}^1(G)$. We note that this is equivalent to having the action of right translation

$$s \mapsto s \cdot f, \text{ where } s \cdot f(t) = f(ts) \text{ for almost every } t \tag{2.4}$$

continuous on G for any fixed f in $\mathrm{S}^1(G)$. We also recall, by well known technique (see [15, pps. 26-27], for example), that any Banach space $\mathcal{V}$ is an contractive essential left/right $\mathrm{L}^1(G)$-module if and only if there is a continuous action/anti-action of G on $\mathcal{V}$ by linear isometries. Furthermore, if $\mathcal{V}$ is an operator space, $\mathcal{V}$ is a completely contractive $\mathrm{L}^1(G)$-module, if and only if the associated action of G on $\mathcal{V}$ is one by complete isometries.

It will be useful, below, to recall the *Lebesgue-Fourier algebra*

$$\mathrm{LA}(G) = \mathrm{A}(G) \cap \mathrm{L}^1(G) \tag{2.5}$$

studied in [10,11,9]. This is simultaneously a Segal algebra in $\mathrm{A}(G)$ and in $\mathrm{L}^1(G)$. In [9] it was shown that $\mathrm{LA}(G)$ is also a contractive operator Segal algebra in either context.

Corollary 2.4 (i) $\mathrm{S}_0(G)$ *is a pseudo-symmetric operator Segal algebra in* $\mathrm{L}^1(G)$*. It is symmetric only if* G *is unimodular.*

(ii) *Let* $\mathcal{I}$ *be a fixed ideal in* $\mathrm{A}(G)$ *satisfying the assumptions of Corollary 2.3. Let* $q'_{\mathcal{I}} : \mathrm{L}^1(G) \hat{\otimes} \mathcal{I} \to \mathrm{A}(G)$ *be given, on elementary tensors, by* $q'_{\mathcal{I}}(f \otimes u) =$

$f*u$. Then $\operatorname{ran} q'_{\mathcal{I}}$, with its quotient operator space structure, is completely isomorphic to $\mathrm{S}_0(G)$.

Proof. (i) If $\mathcal{I}$ is a closed compactly supported pointwise ideal in $\mathrm{LA}(G)$. Then $\mathcal{I}$ imbeds completely contractively into $\mathrm{L}^1(G)$, and is a completely contractive $\mathrm{A}(G)$-module. Thus, it follows Corollary 2.3 above, that $\operatorname{ran} q_{\mathcal{I}}$, with its quotient operator space structure, imbeds completely contractively into $\mathrm{L}^1(G)$. It then follows part (ii) of Theorem 2.2 that $\mathrm{S}_0(G)$ is an operator Segal algebra in $\mathrm{L}^1(G)$; and the same proof of can be trivially adapted to see G acts continuously and completely isometrically on $\mathrm{S}_0(G)$ by right translation. It is clear that $s\cdot u = \Delta(s)u*s^{-1}$ for all u in $\mathrm{S}_0(G)$ and s in G, and hence $\mathrm{S}_0(G)$ is a symmetric Segal algebra in $\mathrm{L}^1(G)$, if and only if G is unimodular.

(ii) Let (e_U) be the bounded approximate identity for $\mathrm{L}^1(G)$ given by normalised indicator functions of relatively compact neighbourhoods of the identity, e. If $[u_{ij}] \in \mathrm{T}_\infty(\mathcal{I})$, then for s in G we have that

$$[s*u_{ij}] = \lim_{U\searrow e}[s*e_U*u_{ij}] \in \mathrm{T}_\infty(\operatorname{ran} q'_{\mathcal{I}}).$$

Thus we see from Lemma 2.1, that $\mathrm{S}_0(G) \subset \operatorname{ran} q'_{\mathcal{I}}$, completely boundedly. To obtain the converse inclusion, we note, similarly as in the proof of Lemma 2.1, that

$$\mathrm{T}_\infty(\mathrm{L}^1(G)\hat{\otimes}\mathcal{I}) \cong \mathrm{L}^1(G)\hat{\otimes}\mathrm{T}_\infty(\mathcal{I}) = \mathrm{L}^1(G) \otimes^\gamma \mathrm{T}_\infty(\mathcal{I}).$$

Hence every element of $\operatorname{ran} q'_{\mathcal{I}}$ is of the form

$$\sum_{k=1}^\infty [f_k*u_{ij}^{(k)}] \text{ where } \sum_{k=1}^\infty \|f_k\|_{\mathrm{L}^1} \left\|[u_{ij}^{(k)}]\right\|_{\mathrm{T}_\infty(\mathrm{A})} < +\infty. \tag{2.6}$$

Now if $[u_{ij}] \in \mathrm{T}_\infty(\mathcal{I})$ and $f \in \mathrm{L}^1(G)$, then

$$[f*u_{ij}] = \int_G f(s)[s*u_{ij}] \in \mathrm{T}_\infty(\mathrm{S}_0(G))$$

since the integral may be realised as a Bochner integral in $\mathrm{T}_\infty(\mathrm{S}_0(G))$, by the continuity of $s \mapsto [s*u_{ij}] : G \to \mathrm{T}_\infty(\mathrm{S}_0(G))$. Thus (2.6) shows that $\operatorname{ran} q'_{\mathcal{I}} \subset \mathrm{S}_0(G)$, completely boundedly. □

The next result shows the only occasions for which we know that $\mathrm{S}_0(G) = \mathrm{LA}(G)$, and we conjecture these are all such occasions. For this result, we will consider $\mathrm{S}_0(G)$ as a Segal algebra in $\mathrm{A}(G)$.

Corollary 2.5 *If K is an open compact subgroup, with T a transversal for left cosets, then there is a natural completely isomorphic algebra homomorphism*

$$\mathrm{S}_0(G) \cong \ell^1(T)\hat{\otimes}\mathrm{A}(K)$$

where $\ell^1(T)$ has pointwise multiplication. In particular,

(i) $\mathrm{S}_0(G) = \ell^1(G)$, *completely isomorphically, if G is discrete, and*

(ii) $\mathrm{S}_0(G) = \mathrm{A}(G)$, *completely isomorphically, if G is compact.*

Proof. We first note that $\mathrm{A}(K) \cong \mathrm{A}_K(G)$, completely isometrically. Second, for s, t in G, $s{*}\mathrm{A}_K(G) = \mathrm{A}_{sK}(G)$ and $t{*}\mathrm{A}_K(G) = \mathrm{A}_{tK}(G)$ are either identical or disjoint, depending on whether $s^{-1}t \in K$ or not. Thus we can see that $q_{\mathrm{A}_K(G)}|_{\ell^1(T)\hat{\otimes}\mathrm{A}_K(G)} : \ell^1(T)\hat{\otimes}\mathrm{A}_K(G) \to \operatorname{ran} q_{\mathrm{A}_K(G)}$ is a complete isomorphism. Indeed, it follows from Lemma 2.1 that

$$\mathrm{T}_\infty(q_{\mathrm{A}_K(G)}|_{\ell^1(T)\hat{\otimes}\mathrm{A}_K(G)}) : \mathrm{T}_\infty(\ell^1(T)\hat{\otimes}\mathrm{A}(K)) \to \mathrm{T}_\infty(\mathrm{S}_0(G))$$

is an bijection, hence an isomorphism. $\square$

It is useful to observe the more general fact below.

Corollary 2.6 *If H is an open subgroup of G, with T a transversal for left cosets, then there is a natural completely isomorphic algebra homomorphism*

$$\mathrm{S}_0(G) \cong \ell^1(T)\hat{\otimes}\mathrm{S}_0(H)$$

where $\ell^1(T)$ has pointwise multiplication.

Proof. Let $\mathcal{I}$ be a non-empty closed compactly supported ideal of $\mathrm{A}(G)$ for which $\mathrm{supp}\mathcal{I} \subset H$. Then we may consider $\mathcal{I}$ to be an ideal in $\mathrm{A}(H) \cong \mathrm{A}_H(G)$, and we have that $q^H_{\mathcal{I}} = q_{\mathcal{I}}|_{\ell^1(H)\hat{\otimes}\mathcal{I}} : \ell^1(H)\hat{\otimes}\mathcal{I} \to \mathrm{S}_0(H)$ is a complete surjection. The bijection $(t, s) \mapsto ts : T{\times}H \to G$ induces an isomorphism

$$\ell^1(T)\hat{\otimes}\ell^1(H) = \ell^1(T) \otimes^\gamma \ell^1(H) \cong \ell^1(T{\times}H) \cong \ell^1(G).$$

Then we obtain the following commuting diagram.

$$\begin{tikzcd}
\ell^1(T)\hat{\otimes}\ell^1(H)\hat{\otimes}\mathcal{I} \arrow[r, "\cong"] \arrow[d, "\mathrm{id}\otimes q^H_{\mathcal{I}}"'] & \ell^1(G)\hat{\otimes}\mathcal{I} \arrow[d, "q_{\mathcal{I}}"] \\
\ell^1(T)\hat{\otimes}\mathrm{S}_0(H) \arrow[r, "\delta_t\otimes u\mapsto t*u"] & \mathrm{S}_0(G)
\end{tikzcd}$$

We obtain, as in the proof of the result above, that the bottom arrow represents a complete isomorphism. $\square$

3 Functorial Properties

3.1 Tensor products

Let us first note the primary motivation for desiring an operator space structure on $\mathrm{S}_0(G)$. This is an analogue of a result from [3] which states that

$$\mathrm{A}(G)\hat{\otimes}\mathrm{A}(H) \cong \mathrm{A}(G\times H)$$

completely isometrically, via the natural morphism which identifies $u\otimes v$ with the function $(s,t)\mapsto u(s)v(t)$. Alternatively we may view this as an analogue of the classical result that

$$\mathrm{L}^1(G)\hat{\otimes}\mathrm{L}^1(H) = \mathrm{L}^1(G)\otimes^\gamma \mathrm{L}^1(H) \cong \mathrm{L}^1(G\times H)$$

where the projective and operator projective tensor products agree since $\mathrm{L}^1(G)$ (or $\mathrm{L}^1(H)$) is a maximal operator space.

Theorem 3.1 *Let G and H be locally compact groups. Then there is a natural complete isomorphism $\mathrm{S}_0(G)\hat{\otimes}\mathrm{S}_0(H) \cong \mathrm{S}_0(G\times H)$.*

Proof. By [14, (7.3)&(7.4)], there are almost connected open subgroups G_0 of G, and H_0 of H. Let $\mathcal{I}$ and $\mathcal{J}$ be compactly supported ideals of $\mathrm{A}(G_0)$ and $\mathrm{A}(H_0)$, respectively. The dual spaces $\mathrm{A}(G_0)^* \cong \mathrm{VN}(G_0)$ and $\mathrm{A}(H_0)^* \cong \mathrm{VN}(H_0)$ are injective von Neumann algebras and hence injective operator spaces; see [21, pps. 227-228], for example. Thus, by comments in [4, p. 130], the inclusion maps induce a complete isometry $\mathcal{I}\hat{\otimes}\mathcal{J} \hookrightarrow \mathrm{A}(G_0)\hat{\otimes}\mathrm{A}(H_0)$. Thus, via the completely isometric injections

$$\mathcal{I}\hat{\otimes}\mathcal{J} \hookrightarrow \mathrm{A}(G_0)\hat{\otimes}\mathrm{A}(H_0) \cong \mathrm{A}(G_0\times H_0) \hookrightarrow \mathrm{A}(G\times H)$$

we may regard $\mathcal{I}\hat{\otimes}\mathcal{J}$ as a compactly supported closed ideal of $\mathrm{A}(G\times H)$.

Now we have a completely isometric identification

$$J : \left(\ell^1(G)\hat{\otimes}\mathcal{I}\right)\hat{\otimes}\left(\ell^1(H)\hat{\otimes}\mathcal{J}\right) \to \ell^1(G\times H)\hat{\otimes}(\mathcal{I}\hat{\otimes}\mathcal{J}).$$

Using finite sums of elementary tensors we see that $q_{\mathcal{I}}\otimes q_{\mathcal{J}} = q_{\mathcal{I}\hat{\otimes}\mathcal{J}}\circ J$. Hence $\mathrm{ran}\,(q_{\mathcal{I}}\otimes q_{\mathcal{J}})$, with its quotient operator space structure, must be completely (isometrically) isomorphic to $\mathrm{ran}\, q_{\mathcal{I}\hat{\otimes}\mathcal{J}}$, with its quotient operator space structure. By projectivity of the operator projective tensor product, see [4, 7.1.7], we have that

$$\mathrm{ran}\,(q_{\mathcal{I}}\otimes q_{\mathcal{J}}) = \mathrm{S}_0(G)\hat{\otimes}\mathrm{S}_0(H).$$

By Corollary 2.3 we have that $\mathrm{ran}\, q_{\mathcal{I}\hat{\otimes}\mathcal{J}} = \mathrm{S}_0(G\times H)$. $\square$

If G and H are abelian, the following recovers one of the main results of Feichtinger [7].

Corollary 3.2 *If either G or H admits an open abelian subgroup, then we have a natural isomorphism* $\mathrm{S}_0(G) \otimes^\gamma \mathrm{S}_0(H) \cong \mathrm{S}_0(G\times H)$.

Proof. If either G or H is abelian, then the proof above can be followed almost verbatim, with the projective tensor product $\otimes^\gamma$ playing the role of the operator projective tensor product $\hat{\otimes}$. The reason we require extra hypotheses here is that they are sufficient (and almost necessary) to obtain that $\mathrm{A}(G) \otimes^\gamma \mathrm{A}(H) \cong \mathrm{A}(G\times H)$, isomorphically, as proved in [18], for example. Thus we obtain that $\mathcal{I} \otimes^\gamma \mathcal{J}$ can be realised as an ideal in $\mathrm{A}(G\times H)$.

If G, say, has an open abelian subgroup A, with transversal for left cosets T, then by Corollary 2.6 and the reasoning above, we obtain isomorphic identifications

$$\begin{aligned}\mathrm{S}_0(G) \otimes^\gamma \mathrm{S}_0(H) &\cong \ell^1(T) \otimes^\gamma \mathrm{S}_0(A) \otimes^\gamma \mathrm{S}_0(H) \\ &\cong \ell^1(T\times\{e_H\}) \otimes^\gamma \mathrm{S}_0(A\times H) \cong \mathrm{S}_0(G\times H)\end{aligned}$$

where we obtain the last identification by realising $T\times\{e_H\}$ as a transversal for the left cosets of $A\times H$ in $G\times H$. □

We note that if G and H are both compact, neither having an open abelian subgroup, then the above result fails by Corollary 2.5 (ii) and [18]. We conjecture that our operator space structure on $\mathrm{S}_0(G)$ is the maximal operator space structure exactly when G admits an open abelian subgroup. This would imply the result above. However, it is clear, only when G has a compact abelian open subgroup, that our operator space structure on $\mathrm{S}_0(G)$ is the maximal one. Indeed if G is abelian, then for an arbitrary closed ideal $\mathcal{I}$ of $\mathrm{A}(G)$, i.e. of $\mathrm{L}^1(\hat{G})$, it is not clear that the subspace operator space structure is the maximal one, whence we have no means to deduce that $\mathrm{S}_0(G) \cong \ell^1(G)\hat{\otimes}\mathcal{I}/\ker q_{\mathcal{I}}$ is a maximal operator space. For results on subspaces of maximal operator spaces, see [20], for example.

3.2 Restriction

We recall from [12,25] that if H is a closed subgroup of G, then the restriction map $u \mapsto u|_H : \mathrm{A}(G) \to \mathrm{A}(H)$ is a quotient map. In fact, it is a complete quotient map since its adjoint map is an injective $*$-homomorphism from $\mathrm{VN}(H)$ onto the von Neumann algebra generated by $\{\lambda(s) : s \in H\}$ in $\mathrm{VN}(G)$.

The following result is due to Feichtinger [7], in the abelian case. However, most of his techniques rely on commutativity of G, and cannot be adapted

to show the general case, even with no considerations for the operator space structure.

Theorem 3.3 *If H is a closed subgroup in G, then the restriction map*

$$u \mapsto u|_H : \mathrm{S}_0(G) \to \mathrm{S}_0(H)$$

is completely surjective.

Proof. First we must verify that if $u \in \mathrm{S}_0(G)$, then $u|_H \in \mathrm{S}_0(H)$. Let T be a transversal for the right cosets of H. The bijection $(t, s) \mapsto st : T \times H \to G$ induces an isomorphism

$$\ell^1(T)\hat{\otimes}\ell^1(H) = \ell^1(T) \otimes^\gamma \ell^1(H) \cong \ell^1(T \times H) \cong \ell^1(G).$$

Now let $\mathcal{I} = \mathrm{A}_K(G)$, where K is a compact neighbourhood of the identity in G. If $t \in T$, then

$$(tK) \cap H \subset s_t(K^{-1}K \cap H), \text{ for some } s_t \text{ in } H. \tag{3.1}$$

Indeed, if $(tK) \cap H \neq \varnothing$, then there is $k \in K$ so $tk \in H$, so $t \in Hk^{-1} \subset HK^{-1}$, and thus there is s_t in H so $t \in s_tK^{-1}$, whence $tk \in s_tK^{-1}K$. Now, as in the proof of Lemma 2.1, any u in $\mathrm{S}_0(G)$ can be written in the form

$$u = \sum_{t \in T} \sum_{s \in H} s{*}t{*}u_{st}, \text{ where } \sum_{t \in T} \sum_{s \in H} \|u_{st}\|_{\mathrm{A}} < +\infty$$

and each $u_{st} \in \mathrm{A}_K(G)$. We then have that for each t in T, using (3.1), that

$$\sum_{s \in H} (s{*}t{*}u_{st})|_H = \sum_{s \in H} s{*}s_t{*}\Big((s_t^{-1}{*}t{*}u_{st})|_H\Big)$$

where, $(s_t^{-1}{*}t{*}u_{st})|_H \in \mathcal{K} = \mathrm{A}_{K^{-1}K \cap H}(H)$ and $\left\|(s_t^{-1}{*}t{*}u_{st})|_H\right\|_{\mathrm{A}} \leq \|u_{st}\|_{\mathrm{A}}$. It then follows that $u|_H$, being a $\|\cdot\|_{\mathrm{ran}\, q_{\mathcal{K}}}$-summable series of elements from $\mathrm{S}_0(H)$, is itself in $\mathrm{S}_0(H)$.

Now let us see that restriction is completely surjective. Let $\mathcal{I}$ be as above so that $(\mathrm{supp}\mathcal{I})^\circ \cap H \neq \varnothing$. Note that $\mathcal{I}|_H$, with the operator space structure given by its being a quotient of $\mathcal{I}$ via the restriction map, is a completely contractive $\mathrm{A}(H)$-module. Indeed, this follows from the fact that $\mathrm{A}(H)$ is a complete quotient of $\mathrm{A}(G)$. Since $\ell^1(H)$ is a (completely) complemented subspace of $\ell^1(G)$, we have that $\ell^1(H)\hat{\otimes}\mathcal{I}$ is a closed subspace of $\ell^1(G)\hat{\otimes}\mathcal{I}$. We have that the following diagram commutes

$$\begin{tikzcd}
\mathrm{T}_\infty(\ell^1(H)\hat{\otimes}\mathcal{I}) \arrow[r, "\mathrm{T}_\infty(\mathrm{id}\otimes(u\mapsto u|_H))"] \arrow[d, "\mathrm{T}_\infty(q_{\mathcal{I}}|_{\ell^1(H)\hat{\otimes}\mathcal{I}})"'] & \mathrm{T}_\infty(\ell^1(H)\hat{\otimes}\mathcal{I}|_H) \arrow[d, "\mathrm{T}_\infty(q_{\mathcal{I}|_H})"] \\
\mathrm{T}_\infty(\mathrm{S}_0(G)) \arrow[r, "{[u_{ij}]\mapsto[u_{ij}|_H]}"] & \mathrm{T}_\infty(\mathrm{S}_0(H))
\end{tikzcd}$$

where $q_{\mathcal{I}|_H}$ is a complete surjection by Corollary 2.3, so $\mathrm{T}_\infty(q_{\mathcal{I}|_H})$ is a surjection by Corollary 1.2. Thus the restriction map $u \mapsto u|_H : \mathrm{S}_0(G) \to \mathrm{S}_0(H)$ is completely surjective. $\square$

3.3 Multipliers on $\overline{\mathrm{L}^2(G)} \otimes^\gamma \mathrm{L}^2(G)$

The aim of this section is to develop some techniques for use in the next section on the averaging operation. Let

$$\mathrm{T}(G) = \overline{\mathrm{L}^2(G)} \otimes^\gamma \mathrm{L}^2(G)$$

where $\overline{\mathrm{L}^2(G)}$ denotes the conjugate space of $\mathrm{L}^2(G)$. We recall that $\mathrm{T}(G)^* \cong \mathcal{B}(\mathrm{L}^2(G))$ via the dual pairing $\left\langle \bar{f} \otimes g, T \right\rangle = \langle Tg|f\rangle$. Thus $\mathrm{T}(G)$ is an operator space with the predual operator space structure.

We may regard $\mathrm{T}(G)$ as a space of equivalence classes of functions on $G\times G$: if $\omega = \sum_{k=1}^\infty \bar{f}_k \otimes g_k$, where $\{f_k\}_{k=1}^\infty$ and $\{g_k\}_{k=1}^\infty$ are each summable sequences from $\mathrm{L}^2(G)$, then $\omega(s,t) = \sum_{k=1}^\infty \overline{f_i(s)} g_i(t)$ for almost every s and almost every t in G. A function $w : G\times G \to \mathbb{C}$ is called a *multiplier* of $\mathrm{T}(G)$ if for every ω in $\mathrm{T}(G)$, $m_w\omega$, defined for almost every s and almost every t in G by $m_w\omega(s,t) = w(s,t)\omega(s,t)$, determines an element of $\mathrm{T}(G)$. We let $\mathrm{MT}(G)$ denote the space of all multipliers w such that $m_w : \mathrm{T}(G) \to \mathrm{T}(G)$ is a bounded map. We summarise below, some results from [24]. We note that there are some trivial differences between our notations used here, and those in [24]; the result is stated below to be consistent with our present notation. For a different perspective, we also refer the reader to [19].

Theorem 3.4 **(i)** *[24, Theo. 3.3] For each w in $\mathrm{MT}(G)$, $m_w : \mathrm{T}(G) \to \mathrm{T}(G)$ is a completely bounded map with $\|m_w\|_{\mathcal{CB}(\mathrm{T}(G))} = \|m_w\|_{\mathcal{B}(\mathrm{T}(G))}$. Thus the space $\mathrm{MT}(G) \cong \{m_w : w \in \mathrm{MT}(G)\}$ is a closed subalgebra of $\mathcal{CB}(\mathrm{T}(G))$, and hence a completely contractive Banach algebra.*

(ii) *[24, Cor. 4.3 & Theo. 5.3] If $u \in \mathrm{B}(G)$, the map $\gamma u : G\times G \to \mathbb{C}$, given by $\gamma u(s,t) = u(st^{-1})$, is an element of $\mathrm{MT}(G)$. Moreover, $\gamma : \mathrm{B}(G) \to \mathrm{MT}(G)$ is a completely contractive homomorphism.*

We let $P_G : \mathrm{T}(G) \to \mathrm{A}(G)$ be given by

$$P_G(\bar{f} \otimes g) = \langle \lambda(\cdot)g|f\rangle = \bar{f} * \check{g} \tag{3.2}$$

where $\check{g}(s) = g(s^{-1})$ for almost every s. The adjoint map, $P_G^* : \mathrm{VN}(G) \to \mathcal{B}(\mathrm{L}^2(G))$, is the inclusion map, hence a complete isometry, whence P_G is a

complete contraction. We note for ω in $\mathrm{T}(G)$ that

$$P_G\omega(s) = \int_G \omega(t, s^{-1}t)dt$$

for each s in g. Thus if $u \in \mathrm{B}(G)$, then

$$P_G(m_{\gamma u}\omega)(s) = \int_G u\Big(t(s^{-1}t)^{-1}\Big)\omega(t, s^{-1}t)dt = u(s)P_G\omega(s).$$

We now introduce a class of ideals in $\mathrm{A}(G)$ which will prove useful. Let K be a compact subset of G of positive measure. Let $\mathrm{T}(K) = \overline{\mathrm{L}^2(K)} \otimes^\gamma \mathrm{L}^2(K)$, where we regard $\mathrm{L}^2(K)$ as a subspace of $\mathrm{L}^2(G)$ in the natural way. We define

$$\mathcal{M}(K) = P_G(\mathrm{T}(K))$$

and endow $\mathcal{M}(K)$ with the quotient operator space structure so it is isometrically isomorphic with $\mathrm{T}(K)/\ker(P_G|_{\mathrm{T}(K)})$. Clearly, $\mathcal{M}(K) \subset \mathrm{A}(G)$, and has as a dense subspace $\mathrm{span}\{\bar{f} * \check{g} : f, g \in \mathrm{L}^2(K)\}$.

Proposition 3.5 *The space $\mathcal{M}(K)$ is a completely contractive $\mathrm{B}(G)$-module. Thus it is an ideal in $\mathrm{A}(G)$ with $\mathrm{supp}\mathcal{M}(K) \subset K^{-1}K$, and equipped with an operator space structure by which it is a completely contractive $\mathrm{A}(G)$-module.*

We remark that there is no reason to suspect that $\mathcal{M}(K)$ is a closed ideal in $\mathrm{A}(G)$ for a general compact set K.

Proof. Since $\mathrm{L}^2(K)$ is a complemented subspace of $\mathrm{L}^2(G)$, $\mathrm{T}(K)$ identifies isometrically as a closed subspace of $\mathrm{T}(G)$. As such, $\mathrm{T}(K)$ is a $\mathrm{MT}(G)$-submodule of $\mathrm{T}(G)$. Also, for u in $\mathrm{B}(G)$ and ω in $\mathrm{T}(K)$ we have $uP_G\omega = P_G(m_{\gamma u}\omega$, so $\mathcal{M}(K)$ is a $\mathrm{B}(G)$-module. We thus have that the following diagram commutes.

$$\begin{tikzcd}
\mathrm{B}(G) \otimes_\wedge \mathrm{T}(K) \arrow[r, "\gamma\otimes\mathrm{id}"] \arrow[d, "\mathrm{id}\otimes P_G|_{\mathrm{T}(K)}"'] & \mathrm{MT}(G)\hat{\otimes}\mathrm{T}(K) \arrow[r, "w\otimes\omega\mapsto m_w\omega"] & \mathrm{T}(K) \arrow[d, "P_G|_{\mathrm{T}(K)}"] \\
\mathrm{B}(G) \otimes_\wedge \mathcal{M}(K) \arrow[rr, "u\otimes v\mapsto uv"] & & \mathcal{M}(K)
\end{tikzcd}$$

Since $\mathrm{id} \otimes P_G|_{\mathrm{T}(K)}$ is a complete quotient map, and the maps $\gamma \otimes \mathrm{id}$, $w \otimes \omega \mapsto m_w\omega$ and $P_G|_{\mathrm{T}(K)}$ are complete contractions, $u \otimes v \mapsto uv$ must be a complete contraction too.

Thus it is clear that $\mathcal{M}(K)$ is an ideal in $\mathrm{A}(G)$ and a completely contractive $\mathrm{A}(G)$-module. It is straightforward to verify that $\mathrm{supp}\mathcal{M}(K) \subset K^{-1}K$. $\square$

Let us note that we may obtain a weak version of a "tensor product factorisation" result of [7]. Let K be a compact subset of G of non-empty interior, and

define $q_K^2 : \ell^1(G) \otimes^\gamma \mathrm{L}^2(K) \to \mathrm{L}^2(G)$ by $q_K^2(\delta_s \otimes f) = s{*}f$. Let $\mathrm{W}^2(G) = \operatorname{ran} q_K^2$ and norm it as the quotient space. It is straightforward to verify that for any other compact set K', having non-empty interior, that $\operatorname{ran} q_{K'}^2 = \operatorname{ran} q_K^2$, and that the quotient norms are equivalent. Let $P'_G : \mathrm{W}^2(G) \otimes^\gamma \mathrm{W}^2(G) \to \mathrm{A}(G)$ be given by $P'_G(f \otimes g) = f{*}\check{g}$. It can be checked, similarly as in the proof of the corollary above, that $\operatorname{ran} P'_G$, with its quotient norm, is a Segal algebra in $\mathrm{A}(G)$ on which G acts isometrically by left translations. Also, $\operatorname{ran} P'_G \subset \mathrm{S}_0(G)$, and thus, by Theorem 2.2 (ii), we obtain $\operatorname{ran} P'_G = \mathrm{S}_0(G)$.

However, unlike in the commutative case, we do not have that either of the maps from $\mathrm{S}_0(G) \otimes^\gamma \mathrm{S}_0(G)$ to $\mathrm{S}_0(G)$, given on elementary tensors by $u \otimes v \mapsto u{*}v$ or $u \otimes v \mapsto u{*}\check{v}$, are surjective. Indeed, this fails for compact groups which do not admit an abelian subgroup of finite index, a fact which follows from [16, Prop. 2.5], in light of Corollary 2.5 and the fact that $v \mapsto \check{v}$ is an isometry on $\mathrm{A}(G)$. It would be interesting to know when either of the aforementioned maps, extended to $\mathrm{S}_0(G)\hat{\otimes}\mathrm{S}_0(G)$, surjects onto $\mathrm{S}_0(G)$. This fails in general. Recent work of the author, with B.E. Forrest and E. Samei, has shown that if G is compact, and hence $\mathrm{S}_0(G) = \mathrm{A}(G)$, then each such map is surjective only when G admits an abelian subgroup of finite index.

3.4 Averaging over a normal subgroup

Let N be a closed normal subgroup of G and $\tau_N : \mathrm{L}^1(G) \to \mathrm{L}^1(G/N)$ be given for f in $\mathrm{L}^1(G)$ and almost every $sN \in G/N$, accepting a mild abuse of notation, by

$$\tau_N(f)(sN) = \int_N f(sn)dn.$$

This operator is a complete quotient map as observed in [9]. It was shown in [7], for an abelian G, that $\tau_N(\mathrm{S}_0(G)) = \mathrm{S}_0(G/N)$. We obtain a generalisation of that result.

Theorem 3.6 *We have for any locally compact group G with closed normal subgroup N that $\tau_N(\mathrm{S}_0(G)) = \mathrm{S}_0(G/N)$, and $\tau_N : \mathrm{S}_0(G) \to \mathrm{S}_0(G/N)$ is a complete surjection.*

Proof. We divide the proof into three stages.

(I) $\tau_N(\mathcal{M}(K)) \subset \mathrm{A}(G/N)$ *and* $\tau_N : \mathcal{M}(K) \to \mathrm{A}(G/N)$ *is completely bounded.*

Let us first show that $\tau_N(\mathrm{L}^2(K)) \subset \mathrm{L}^2(G/N)$ and that $\tau_N : \mathrm{L}^2(K) \to \mathrm{L}^2(G/N)$ is bounded. Let $\varphi : G \to \mathbb{C}$ be a continuous function of compact support such that $\varphi|_K = 1$. Then for any f in $\mathrm{L}^2(K)$ we have, using Hölder's inequality and

the Weyl integral formula, that

$$\begin{aligned}
\|\tau_N(f)\|^2_{\mathrm{L}^2(G/N)} &= \int_{G/N}\left|\int_N \varphi(sn)f(sn)dn\right|^2 dsN \\
&\leq \int_{G/N}\int_N |\varphi(sn')|^2dn'\int_N |f(sn)|^2dn\, dsN \\
&\leq \sup_{s\in G}\tau_N(|\varphi|^2)(sN)\int_G |f(s)|^2ds.
\end{aligned}$$

We note that $\sup_{s\in G}\tau_N(|\varphi|^2)(sN)<\infty$ since $\tau_N(|\varphi|^2)$ is itself continuous and of compact support on G/N, as can be checked using the uniform continuity of $|\varphi|^2$. Thus

$$\|\tau_N\|_{\mathcal{B}(\mathrm{L}^2(K),\mathrm{L}^2(G/N))}\leq \sup_{s\in G}\tau_N(|\varphi|^2)(sN)^{1/2}.$$

Is is shown in [17, p. 187] that for any compactly supported $f\in \mathrm{L}^1(G)$ that

$$\tau_N(\check{f}) = [\Delta_{G/N}\tau_N(\check{\Delta}_G f)]^\vee$$

Now let $\theta_N : \mathrm{L}^2(K)\to \mathrm{L}^2(G/N)$ be given by

$$\theta_N(f) = \Delta_{G/N}\tau_N(\check{\Delta}_G f).$$

Then we have that θ_N is bounded with

$$\|\theta_N\|_{\mathcal{B}(\mathrm{L}^2(K),\mathrm{L}^2(G/N))}\leq \sup_{s\in K}\Delta_{G/N}(sN)\,\|\tau_N\|_{\mathcal{B}(\mathrm{L}^2(K),\mathrm{L}^2(G/N))}\sup_{t\in K}\frac{1}{\Delta_G(t)}.$$

Now if $f,g\in \mathrm{L}^2(K)$, then

$$\tau_N(\bar{f}*\check{g}) = \tau_N(\bar{f})*\tau_N(\check{g}) = \overline{\tau_N(f)}*[\theta_N(g)]^\vee \in \mathrm{A}(G/N).$$

Hence it follows that $\tau_N(\mathcal{M}(K))\subset \mathrm{A}(G/N)$.

We now wish to establish that $\tau_N:\mathcal{M}(K)\to \mathrm{A}(G/N)$ is completely bounded. As in [4, Sec. 3.4], we assign $\overline{\mathrm{L}^2(K)}$ the row space operator space structure, denoted $\overline{\mathrm{L}^2(K)}_r$; and $\mathrm{L}^2(K)$ the column operator space structure space, denoted $\mathrm{L}^2(K)_c$. Then we have a completely isometric equality

$$\mathrm{T}(G) = \overline{\mathrm{L}^2(K)}_r\hat{\otimes}\mathrm{L}^2(K)_c$$

by [4, 9.3.2 and 9.3.4]. Then, by [4, 3.4.1 and 7.1.3], we have that

$$\tau_N\otimes\theta_N : \overline{\mathrm{L}^2(K)}_r\hat{\otimes}\mathrm{L}^2(K)_c \to \overline{\mathrm{L}^2(G/N)}_r\hat{\otimes}\mathrm{L}^2(G/N)_c$$

is completely bounded, thus is a completely bounded map on $\mathrm{T}(K)$. Hence we see that the following diagram commutes

$$\begin{tikzcd} \mathrm{T}(K) \arrow[r, "\tau_N\otimes\theta_N"] \arrow[d, "P_G|_{\mathrm{T}(K)}"'] & \mathrm{T}(G/N) \arrow[d, "P_{G/N}"] \\ \mathcal{M}(K) \arrow[r, "\tau_N"] & \mathrm{A}(G/N) \end{tikzcd}$$

where P_G is defined in (3.2), and $P_{G/N}$ is defined analogously. Since $P_G|_{\mathrm{T}(K)}$ is a complete quotient, and $\tau_N \otimes \theta_N$ and $P_{G/N}$ are completely bounded, $\tau_N : \mathcal{M}(K) \to \mathrm{A}(G/N)$ must be completely bounded too.

(II) $\tau_N(\mathcal{M}(K))$*, with its quotient operator space structure, i.e. naturally identified with the quotient space* $\mathcal{M}(K)/\ker(\tau_N|_{\mathcal{M}(K)})$*, is a completely contractive* $\mathrm{A}(G/N)$*-module.*

Let $\pi_N : G \to G/N$ denote the quotient map. By [5, (2.26)] the function $u \mapsto u{\circ}\pi_N$ defines a complete isometry from $\mathrm{B}(G/N)$ to $\mathrm{B}(G\!:\!N)$, the closed subspace of $\mathrm{B}(G)$ of functions which are constant on cosets of N. Now if $u \in \mathrm{B}(G/N)$ and $v \in \mathcal{M}(K)$, then for any s in G we have

$$u(sN)\tau_N(u)(sN) = \int_N u{\circ}\pi_N(sn)v(sn)dn = \tau_N(u{\circ}\pi_N\, v)(sN)$$

Hence $u\tau_N(v) = \tau(u{\circ}\pi_N\, v) \in \tau_N(\mathcal{M}(K))$, by Proposition 3.5.

We now establish that $\tau_N(\mathcal{M}(K))$ is a completely contractive $\mathrm{B}(G/N)$-module, hence a completely contractive $\mathrm{A}(G/N)$-module. Letting $\iota : \mathrm{B}(G : N) \to \mathrm{B}(G/N)$ be the inverse of $u \mapsto u{\circ}\pi_N$, we obtain the following commuting diagram.

$$\begin{tikzcd} \mathrm{B}(G\!:\!N) \otimes_\wedge \mathcal{M}(K) \arrow[r, "u\otimes v\mapsto uv"] \arrow[d, "\iota\otimes\tau_N|_{\mathcal{M}(K)}"'] & \mathcal{M}(K) \arrow[d, "\tau_N|_{\mathcal{M}(K)}"] \\ \mathrm{B}(G/N) \otimes_\wedge \tau_N(\mathcal{M}(K)) \arrow[r, "\tilde{u}\otimes\tilde{v}\mapsto\tilde{u}\tilde{v}"] & \mathrm{A}(G/N) \end{tikzcd}$$

Since $\iota \otimes \tau_N|_{\mathcal{M}(K)}$ is a complete contraction, and $u \otimes v \mapsto uv$ and $\tau_N|_{\mathcal{M}(K)}$ are complete contractions, we obtain that $\tilde{u} \otimes \tilde{v} \mapsto \tilde{u}\tilde{v}$ is thus a complete contraction too.

(III) *The finale.*

We recall from Corollary 2.4 (ii), that the map $q'_{\mathcal{M}(K)} : \mathrm{L}^1(G)\hat{\otimes}\mathcal{M}(K) \to \mathrm{S}_0(G)$, given on elementary tensors by $q'_{\mathcal{M}(K)} f \otimes u = f{*}u$ is a complete surjection. Similarly, appealing also to (II) above, we have that $q'_{\tau_N(\mathcal{M}(K))} : \mathrm{L}^1(G/N)\hat{\otimes}\tau_N(\mathcal{M}(K)) \to \mathrm{S}_0(G/N)$ is a complete surjection. It is then clear

that the following diagram commutes.

$$
\begin{tikzcd}
\mathrm{L}^1(G)\hat{\otimes}\mathcal{M}(K) \arrow[r, "\tau_N\otimes\tau_N"] \arrow[d, "q'_{\mathcal{M}(K)}"'] & \mathrm{L}^1(G/N)\hat{\otimes}\tau_N(\mathcal{M}(K)) \arrow[d, "q'_{\tau_N(\mathcal{M}(K))}"] \\
\mathrm{S}_0(G) \arrow[r, "\tau_N"] & \mathrm{L}^1(G/N)
\end{tikzcd}
$$

In particular $\tau_N(\mathrm{S}_0(G)) \subset \operatorname{ran} q'_{\tau_N(\mathcal{M}(K))} = \mathrm{S}_0(G/N)$. Moreover, since $q'_{\mathcal{M}(K)}$ and $\tau_N \otimes \tau_N$, as above, and $q'_{\tau_N(\mathcal{M}(K))} : \mathrm{L}^1(G/N)\hat{\otimes}\tau_N(\mathcal{M}(K)) \to \mathrm{S}_0(G/N)$ are complete surjections, $\tau_N : \mathrm{S}_0(G) \to \mathrm{S}_0(G/N)$ is completely bounded. Amplifying the diagram by T_∞, and appealing to Corollary 1.2, as in the end of the proof of Theorem 3.3, we see that $\tau_N : \mathrm{S}_0(G) \to \mathrm{S}_0(G/N)$ is completely surjective. $\square$

We remark that it follows from the above theorem that $\tau_N(\mathrm{A}_c(G)) \subset \mathrm{A}_c(G/N)$. We note that it was proved in [9] that $\tau_N(\mathrm{LA}(G)) = \mathrm{L}^1(G/N)$. We also note that it was shown by Lohoué [17] that $\tau_N : \mathrm{A}_K(G) \to \mathrm{A}(G/N)$ is a bounded map, with bounded depending on K. This fact can be deduced from our result, but Lohoué's proof is much simpler, though it is not obvious how to adapt his proof to show that $\tau_N \in \mathcal{CB}(\mathrm{A}_K(G), \mathrm{A}(G/N))$.

3.5 An Isomorphism Theorem

Let G and H be locally compact groups. Wendel [27] proved that there is an isometric isomorphism between the convolution algebras $\mathrm{L}^1(G)$ and $\mathrm{L}^1(H)$ if and only if G and H are isomorphic topological groups [27]. Also, Walter [26] proved that $\mathrm{A}(G)$ and $\mathrm{A}(H)$ are isometrically isomorphic if and only if G and H are isomorphic topological groups. Since we lack fixed norms on our algebras $\mathrm{S}_0(G)$ and $\mathrm{S}_0(H)$, it is not reasonable to expect and "isometric isomorphism" theorem, in the spirit of Wendel's and Walter's theorems. In fact, if G and H are both discrete groups having the same cardinality, then Corollary 2.5 tells us there is a multiplicative isomorphism identifying $\mathrm{S}_0(G) \cong \mathrm{S}_0(H)$. Similarly, if G and H are finite abelian groups, then there is a convolutive isomorphism identifying $\mathrm{S}_0(G) \cong \mathrm{S}_0(H)$. To obtain a satisfactory result, we must simultaneously exploit the facts $\mathrm{S}_0(G)$ and $\mathrm{S}_0(H)$ are pointwise and convolutive algebras.

Theorem 3.7 *Let G and H be locally compact groups and $\Phi : \mathrm{S}_0(G) \to \mathrm{S}_0(H)$ be a bounded linear bijection which satisfies*

$$\Phi(uv) = \Phi u \, \Phi v \quad \textit{and} \quad \Phi(u*v) = \Phi u * \Phi v$$

for every u, v in $\mathrm{S}_0(G)$. There is a homeomorphic isomorphism $\alpha : G \to H$

such that

$$\Phi u = u \circ \alpha$$

for each u in $\mathrm{S}_0(G)$.

Proof. We recall that $\mathrm{A}(G)$ has Gelfand spectrum G, implemented by evaluation functionals. Since $\mathrm{S}_0(G)$ is a Segal algebra in $\mathrm{A}(G)$, it follows from [2, Thm. 2.1] that $\mathrm{S}_0(G)$ has Gelfand spectrum G too. The same holds for $\mathrm{S}_0(H)$. Thus we may define $\alpha : H \to G$ by letting for h in H, $\alpha(h)$ be the element of G which satisfies $u\Big(\alpha(h)\Big) = \Phi u(h)$ for each $u \in \mathrm{S}_0(G)$. Then α is continuous. Indeed, if not, we may find a net $h_i \to h$ in H and a neighbourhood U of $\alpha(h)$, such that $\alpha(h_i) \notin U$ for each i. Using regularity we may find u in $\mathrm{S}_0(G)$ such that $u\Big(\alpha(h)\Big) = 1$ and $\mathrm{supp} u \subset U$. But then we would obtain

$$\lim_i \Phi u(h_i) = \lim_i u\Big(\alpha(h_i)\Big) = 0 \neq 1 = u\Big(\alpha(h)\Big) = \Phi u(h)$$

which contradicts that Φu is continuous, in particular that $\Phi u \in \mathrm{S}_0(H)$. We may similarly obtain a continuous map $\beta : G \to H$ satisfying $v\Big(\beta(s)\Big) = \Phi^{-1}v(s)$ for all $s \in G$ and v in $\mathrm{S}_0(H)$. We clearly have that $\beta = \alpha^{-1}$, hence α is a homeomorphism.

It remains to see that α is a group homomorphism. Let $\mathcal{U}$ be a compact neighbourhood basis of the identity e_G in G. For each U in $\mathcal{U}$ find u_U in $\mathrm{S}_0(G)$ such that

$$\mathrm{supp} u_U \subset U \quad \text{and} \quad \int_G |u_U(s)| ds = 1.$$

Then (u_U) is a bounded approximate identity for $\mathrm{L}^1(G)$, hence a convolutive approximate unit for $\mathrm{S}_0(G)$. Since Φ is a surjective convolutive homomorphism, (Φu_U) is a convolutive approximate identity for $\mathrm{S}_0(H)$. Let h_1, h_2 in H and suppose that $\alpha(h_1)^{-1}\alpha(h_2) \neq \alpha(h_1^{-1}h_2)$. We could then find $v \in \mathrm{S}_0(G)$ such that

$$v\Big(\alpha(h_1)^{-1}\alpha(h_2)\Big) = 1 \text{ and } v(s) = 0 \text{ for all } s \text{ in a nbhd. of } \alpha(h_1^{-1}h_2).$$

Then we would have that

$$\begin{aligned} v\Big(\alpha(h_1)^{-1}\alpha(h_2)\Big) &= \alpha(h_1){*}v\Big(\alpha(h_2)\Big) = \lim_U \alpha(h_1){*}u_U{*}v\Big(\alpha(h_2)\Big) \\ &= \lim_U \Phi\Big(\alpha(h_1){*}u_U{*}v\Big)(h_2) \\ &= \lim_U \Phi\Big(\alpha(h_1){*}u_U\Big){*}\Phi v(h_2) \\ &= \lim_U \int_H u_U\Big(\alpha(h_1)^{-1}\alpha(r)\Big)v\Big(\alpha(r^{-1}h_2)\Big)dr \\ &= 0 \end{aligned}$$

where we obtain the last equality from the fact that

$$\mathrm{supp}\Big(\alpha(h_1){*}u\Big)\circ\alpha \subset \{r \in H : \alpha(h_1)^{-1}\alpha(r) \in U\} = \alpha^{-1}\Big(\alpha(h_1)U\Big)$$

and the supposition that $v\Big(\alpha(r^{-1}h_2)\Big) = 0$ for all r in $\alpha^{-1}\Big(\alpha(h_1)U\Big)$, for a suitably small choice of U. This contradicts that $v\Big(\alpha(h_1)^{-1}\alpha(h_2)\Big) = 1$, whence such a v cannot be chosen, and we thus conclude that $\alpha(h_1)^{-1}\alpha(h_2) = \alpha(h_1^{-1}h_2)$. Substituting $e_H = h_1$, we see that $\alpha(e_H) = e_G$ and then, substituting e_H for h_2, we obtain that $\alpha(h_1^{-1}) = \alpha(h_1)^{-1}$ for each h_1 in H. Thus α is a group homomorphism. □

Our theorem above is not special to the class of algebras $\mathrm{S}_0(G)$. In fact it can be applied to any class of regular Banach algebras with spectrum G, each of which is a Segal algebra of $\mathrm{L}^1(G)$. Examples of such are $\mathrm{LA}(G)$ from (2.5) and the Wiener algebra $\mathrm{W}_0(G)$ as defined in [6].

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